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Localization of the bending response in presence of axial load

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Abstract

In this paper, we study the effect of a small misplacement on the deflection response of the two-span column that is subjected to transverse loading. Whereas the buckling modes of such and other structures have been investigated thoroughly in the literature, the authors tend not to investigate the most important quantity, namely, the column's response. This gap is attempted to be filled by the present study. It is remarkable that a 3% misplacement may lead to a five-fold enhancement of the response, and thus, location imperfections cannot be overlooked for safe design. © 2000 Published by Elsevier Science Ltd.

Keywords: Localization; Buckling load; Buckling modes

1. Introduction

The study of the localization phenomenon in buckling of structures was pioneered by Pierre and Plaut (1989). They uncovered a strong effect of a small misplacement in the location of an intermediate support on the buckling behavior of two-span column with intermediate rotational spring. They demonstrated that the buckling modes are considerably influenced by the presence of the torsional spring. Although the effect of the misplacement turned out to be insignificant on the buckling load itself, the buckling modes were altered considerably, with attendant strong localization, i.e. displacements in the buckling modes in one of the spans being significantly larger than the other. A similar study was conducted by Nayfeh and Hawwa (1999), Nayfeh and Hawwa (1994), who studied three- and four-span columns. The case of N-span columns with a misplacement occurring in a single span was studied by Li et al. (1995). The localization of buckling modes in the trusses was studied by Brasil and Hawwa (1995),

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Luongo and Pignataro (1998). In above studies, the buckling modes were studied extensively. For other localization studies in buckling, the readers may consult with the special issue of the journal "Chaos, Solitons & Fractals" (Xie, 2000) and references quoted therein.

The following consideration is worth mentioning: The buckling modes themselves, although extremely important, constitute only an auxiliary information. They could be utilized for expanding the response quantities in series in terms of these modes. Yet, the study of buckling modes only does not lead to conclusion about the response quantities, the designer may be concerned with.

In this study, we address the effect of small misplacements in the same two-span column as considered by Pierre and Plaut (1989) but now the column is subjected to transverse loading, in addition to axial compressive forces. Significant influence of the misplacements is detected on the response, including, for specific combinations of parameters and in some locations over a five-fold enhancement of the response, for just 3% of the misplacement in the middle support.

2. Structural model and governing equations

Let us consider a two-span column with an elastic torsional spring with stiffness c connected at the middle support. The effect of the spring is represented through its moment furnished at the column's support B and of magnitude

$$M_B = -c\varphi_B \tag{1}$$

where φ_B denotes the slope at the middle support. The column is subjected to axial compressive load P. The middle support of the column is misplaced with respect to its nominally symmetric location (Fig. 1), d denotes the value of the misplacement. The misplacement is considered positive if it occurs to the right of the middle support. The differential equations governing the transverse deflection of the beam, in presence of external transverse load $q(x_i)$ reads

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left[EI(x_1)\frac{\mathrm{d}^2 w_1(x_1)}{\mathrm{d}x_1^2}\right] + P\frac{\mathrm{d}^2 w_1(x_1)}{\mathrm{d}x_1^2} = q(x_1), \quad \text{for } 0 \le x_1 \le \frac{L}{2} + d \tag{2}$$

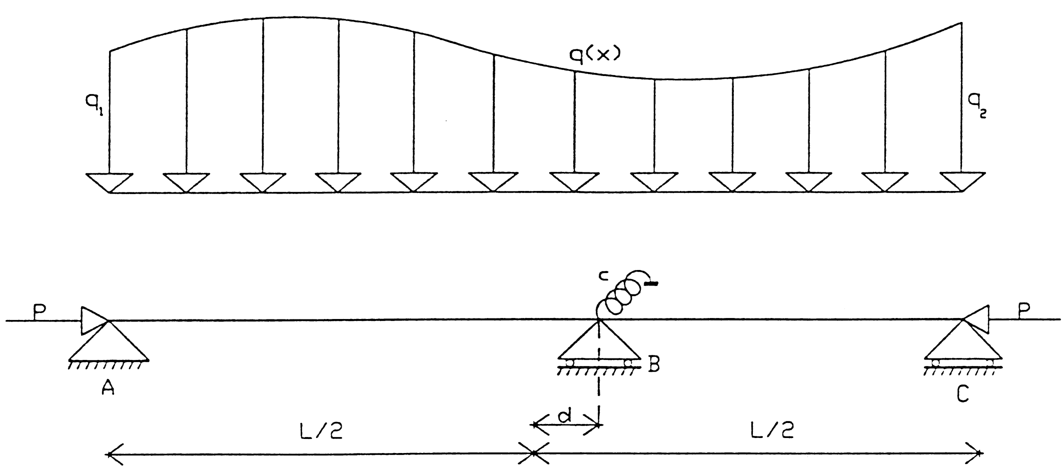

Fig. 1. Structural model.

$$\frac{\mathrm{d}^2}{\mathrm{d}x_2^2}\left[EI(x_2)\frac{\mathrm{d}^2 w_2(x_2)}{\mathrm{d}x_2^2}\right] + P\frac{\mathrm{d}^2 w_2(x_2)}{\mathrm{d}x_2^2} = q(x_2), \quad \text{for } 0 \le x_2 \le \frac{L}{2} - d \tag{3}$$

where the two different coordinate systems $(Oxw)_i$ $(i = 1, 2)$ are adopted. The boundary conditions associated with Eqs. (2) and (3) are

$$w_1(0) = 0; \quad \left.\frac{\mathrm{d}^2 w_1(x_1)}{\mathrm{d}x_1^2}\right|_0 = 0 \tag{4a}$$

$$w_2(L/2 - d) = 0; \quad \left.\frac{\mathrm{d}^2 w_2(x_2)}{\mathrm{d}x_2^2}\right|_{L/2-d} = 0 \tag{4b}$$

representing the boundary conditions at the outer supports A and C. The four continuity conditions, at the middle support B, read

$$w_1(L/2 + d) = 0; \quad w_2(0) = 0 \tag{5a}$$

$$\left.\frac{\mathrm{d}w_1(x_1)}{\mathrm{d}x_1}\right|_{L/2+d} = \left.\frac{\mathrm{d}w_2(x_2)}{\mathrm{d}x_2}\right|_0 \tag{5b}$$

$$EI\left.\frac{\mathrm{d}^2 w_2(x_2)}{\mathrm{d}x_2^2}\right|_0 - EI\left.\frac{\mathrm{d}^2 w_1(x_1)}{\mathrm{d}x_1^2}\right|_{L/2+d} - c\left.\frac{\mathrm{d}w_1(x_1)}{\mathrm{d}x_1}\right|_{L/2+d} = 0 \tag{5c}$$

Eq. (5c) represents the equilibrium between the external moment at the support B and the internal bending moments of the column. In the following developments, it will be convenient to cast Eqs. (2) and (5a)–(5c) in non-dimensional form. We introduce the non-dimensional local coordinates

$$\xi_i = x_i/L, \quad (i = 1, 2) \tag{6}$$

The differential equation governing the transverse deflection of the uniform column, will assume the following form

$$\frac{\mathrm{d}^4 w_1(\xi_1)}{\mathrm{d}\xi_1^4} + \lambda^2\frac{\mathrm{d}^2 w_1(\xi_1)}{\mathrm{d}\xi_1^2} = \bar{q}(\xi_1), \quad \text{for } 0 \le \xi_1 \le \alpha \tag{7}$$

$$\frac{\mathrm{d}^4 w_2(\xi_2)}{\mathrm{d}\xi_2^4} + \lambda^2\frac{\mathrm{d}^2 w_2(\xi_2)}{\mathrm{d}\xi_2^2} = \bar{q}(\xi_2), \quad \text{for } 0 \le \xi_2 \le \beta \tag{8}$$

Quantity $\lambda^2 = PL^2/EI$ is a non-dimensional load parameter, while the non-dimensional lengths α and β are defined as follows

$$\alpha = 1/2 + \delta, \quad \beta = 1/2 - \delta, \quad \delta = d/L \tag{9}$$

where δ is the non-dimensional misplacement. The right-hand side of Eqs. (7) and (8) is the variable non-dimensional load

$$\bar{q} = q(x_i)L^4/EI, \quad (i = 1, 2) \tag{10}$$

It is instructive to first consider the effect of the misplacement on buckling modes. To this end, we fix $\bar{q}(x)$ at zero. The general integrals of Eqs. (7) and (8) with $\bar{q}(x) = 0$ are

$$w_1(\xi_1) = A_1 + B_1\xi_1 + C_1 \cos(\lambda\xi_1) + D_1 \sin(\lambda\xi_1) \tag{11}$$

$$w_2(\xi_2) = A_2 + B_2\xi_2 + C_2 \cos(\lambda\xi_2) + D_2 \sin(\lambda\xi_2) \tag{12}$$

which satisfy the boundary conditions stated in Eqs. (11) and (12). Imposing the boundary conditions at $x = 0$, we get $A_1 = C_1 = 0$, reducing the number of unknowns to six. They satisfy the following equations

$$B_1\,\alpha + D_1 \sin(\lambda\alpha) = 0 \tag{13a}$$

$$C_2 + A_2 = 0 \tag{13b}$$

$$B_2 + D_2\,\lambda - B_1 - D_1\lambda \cos(\lambda\alpha) = 0 \tag{13c}$$

$$-C_2\lambda^2 + D_1\lambda^2 \sin(\lambda\alpha) + \gamma\left[B_1 + D_1\lambda \cos(\lambda\alpha)\right] = 0 \tag{13d}$$

$$A_2 + B_2\beta + C_2 \cos(\lambda\beta) + D_2 \sin(\lambda\beta) = 0 \tag{13e}$$

$$-C_2\lambda^2 \cos(\lambda\beta) - D_2\lambda^2 \sin(\lambda\beta) = 0 \tag{13f}$$

where $\gamma = cL/EI$ represents the ratio between the torsional stiffness c and the column's stiffness EI. Eq. (13d) is the result of substitution of Eqs. (11) and (12) into the following non-dimensional analog of Eq. (5c)

$$\left.\frac{\mathrm{d}^2 w_2(\xi_2)}{\mathrm{d}\xi_2^2}\right|_0 - \left.\frac{\mathrm{d}^2 w_1(\xi_1)}{\mathrm{d}\xi_1^2}\right|_\alpha - \gamma\left.\frac{\mathrm{d}w_1(\xi_1)}{\mathrm{d}\xi_1}\right|_\alpha = 0 \tag{14}$$

Non-triviality conditions leads to the following characteristic equation for the parameter λ

$$\begin{aligned}&\beta\lambda\left(\alpha\lambda^2 + \gamma\right) \sin(\alpha\lambda) \cos(\beta\lambda) - \left(\gamma + \lambda^2\right) \sin(\alpha\lambda)\sin(\beta\lambda) + \alpha\lambda\left(\beta\lambda^2 + \gamma\right) \sin(\beta\lambda) \cos(\alpha\lambda)\\ &\quad - \alpha\beta\gamma\lambda^2 \cos(\alpha\lambda) \cos(\beta\lambda)\\ &\quad = 0\end{aligned} \tag{15}$$

Eq. (15) coincides with Eq. (2) of Pierre and Plaut (1989). Fig. 2 shows the dependence of the non-dimensional buckling load parameter λ_1 on the non-dimensional misplacement δ for different values of the torsional spring ratio γ. For large values of γ, the critical buckling load tends to that of the clamped–simply supported (C–S) column of $L/2$ with buckling load $P_{cl} = 2.046\pi^2 EI/(L/2)^2$, corresponding to $\lambda = \sqrt{8.19\pi^2} = 9.02$. The values $\gamma = 600$ and $\delta = 0$ correspond to the buckling load 8.89 that differs from the buckling load of the C–S column, $\lambda = 9.02$ by 1.22%. For $\gamma \rightarrow \infty$, the spans of the column act independently. This shows that the parameter γ could be interpreted as a parameter of decoupling: The greater γ corresponds to a larger extent of decoupling.

Substitution of the solution $\lambda = \lambda_j$ of Eq. (15) into the algebraic linear system in Eqs. (13a)–(13f) yields a set of six linearly dependent equations in six unknowns. After some algebra, we get, with A_2 set equal to unity, the following expressions

$$B_1 = \frac{\lambda^2 \sin(\lambda\alpha)}{\gamma \sin(\lambda\alpha) + \lambda^2\alpha \sin(\lambda\alpha) - \lambda\gamma\alpha \cos(\lambda\alpha)}$$

$$D_1 = -\frac{\lambda^2 \sin(\lambda\alpha)}{\gamma \sin(\lambda\alpha) + \lambda^2\alpha \sin(\lambda\alpha) - \lambda\gamma\alpha \cos(\lambda\alpha)}$$

$$D_2 = -\frac{\beta\left[B_1 + \lambda D_1 \cos(\lambda\alpha)\right] - \cos(\lambda\beta) + 1}{\sin(\lambda\beta) - \lambda\beta}$$

$$B_2 = B_1 + \lambda D_1 \cos(\lambda\alpha) - \lambda D_2, \quad C_2 = -A_2 = -1 \tag{16}$$

The buckling modes read

$$w_1(\xi_1) = B_1\xi_1 + D_1 \sin(\lambda_j\xi_1), \quad 0 \leq \xi_1 \leq \alpha \tag{17}$$

$$w_2(\xi_2) = 1 - \cos(\lambda_j\xi_2) + B_2\xi_2 + D_2 \sin(\lambda_j\xi_2), \quad 0 \leq \xi_2 \leq \beta \tag{18}$$

The buckling modes for different values of the misplacement in the middle support are portrayed in Fig. 3a and b. Fig. 3a portrays the first buckling mode for misplacement of 1.5%, whereas in Fig. 3b the misplacement constitutes 3%. We observe that for greater values of γ, the buckling modes exhibit

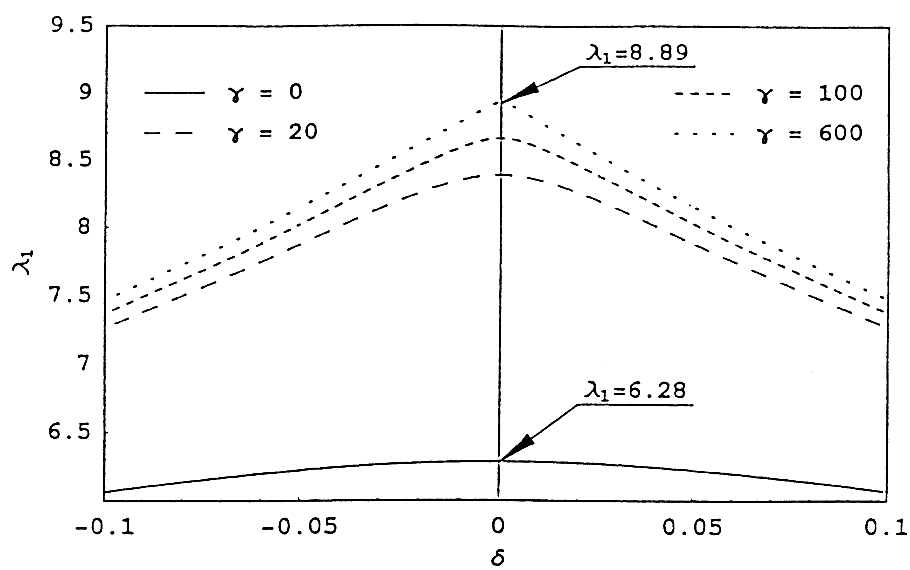

Fig. 2. Influence of the misplacement δ on the first buckling load λ_1 for different values of the coefficient γ.

large differences in the magnitudes of the displacement in the first and second spans. This pattern is an essence of the localization phenomenon (Brasil and Hawwa, 1995; Pierre and Plaut, 1989; Li et al., 1995; Nayfeh and Hawwa, 1994). On the other hand, as Fig. 2 suggests, the influence of misplacements on the classical buckling load is insignificant. The examination of Fig. 2 shows that large misplacements of the order of 7–10% of the total length are needed to change the classical buckling load by 10%. However, by considering the misplacement constituting a 10% of the column length (as done in some determinietic and stochastic misplacement) we, *de facto*, treat another system rather than a slight perturbation of the nominal system. It appears that the localization studies must concentrate on very small misplacements.

Fig. 3a and b show the effect of the torsional stiffness ratio γ on the onset of localization. We observe

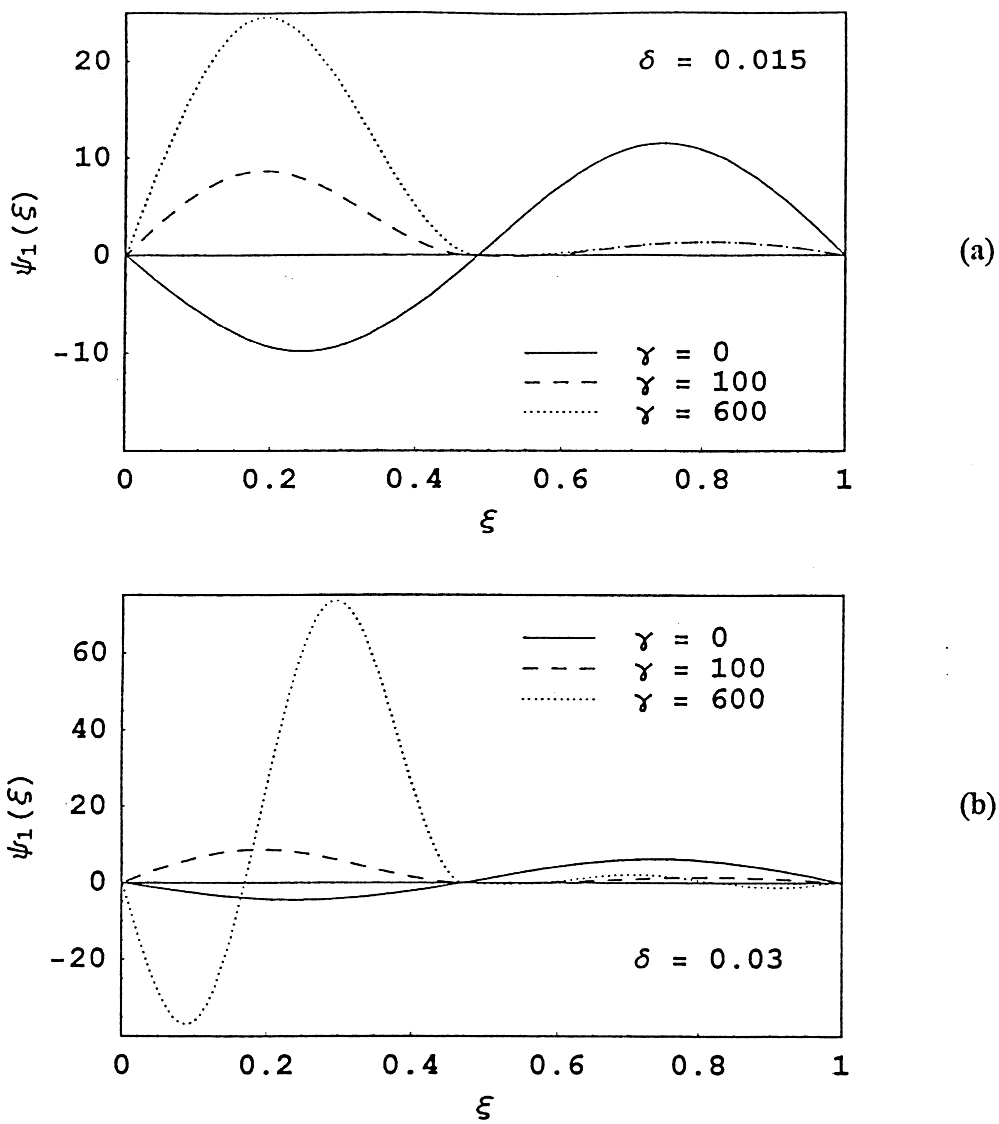

Fig. 3. First buckling mode for different stiffness coefficients γ and misplacements $\delta = 0.015$ (a) and $\delta = 0.003$ (b).

that for small values of the ratio γ the localization effect is absent. A strong localization is found for $\gamma = 600$. Note that the aim of this section was auxiliary, to introduce the necessary equations and consider the buckling modes. The objective of the next sections is to analyze the influence of the misplacement on the transverse response of the column subjected simultaneously to axial and transverse loads.

3. Influence of misplacements on transverse response

Let us consider the two-span column subjected to the simultaneous action of a transverse and axial loads. The distributed load $q_i(x_i)$ is assumed to vary linearly along the beam's axis

$$q_1(x_1) = q_1 - x_1 \tan(\varphi), \quad \text{for } 0 \le x_1 \le L/2 + d \tag{19a}$$

$$q_2(x_2) = q_1 - (L/2 + d + x_2)\tan(\varphi) \quad \text{for } 0 \le x_2 \le L/2 - d \tag{19b}$$

with $\tan(\varphi) = 2q_1/L$, so that it is anti-symmetric with respect to the mid cross-section. We introduce the non-dimensional load $\bar{q}_i = (q_i L^4)/(EI)$ $(i = 1, 2)$. The functions $\bar{q}_i(\xi_i)$ $(i = 1, 2)$ become

$$\bar{q}_1(\xi_1) = \bar{q}_1 - \xi_1 L^5 \tan(\varphi)/EI, \quad \text{for } 0 \le \xi_1 \le \alpha \tag{20a}$$

$$\bar{q}_2(\xi_2) = \bar{q}_1 - (\alpha + \xi_2) L^5 \tan(\varphi)/EI, \quad \text{for } 0 \le \xi_2 \le \beta \tag{20b}$$

The governing equations for the transverse displacement are

$$\frac{d^4 w_1}{d\xi_1^4} + \lambda^2 \frac{d^2 w_1}{d\xi_1^2} = \bar{q}_1 - \xi_1 L^5 \tan(\varphi)/EI, \quad \text{for } 0 \le \xi_1 \le \alpha \tag{21a}$$

$$\frac{d^4 w_2}{d\xi_2^4} + \lambda^2 \frac{d^2 w_2}{d\xi_2^2} = \bar{q}_1 - (\xi_2 + \alpha) L^5 \tan(\varphi)/EI, \quad \text{for } 0 \le \xi_2 \le \beta \tag{21b}$$

For the non-dimensional displacements $y_i(\xi) = w_i(\xi)/L$, we get

$$y_1(\xi_1) = A_1 + B_1\xi_1 + C_1 \cos(\lambda\xi_1) + D_1 \sin(\lambda\xi_1) + \frac{\bar{q}_1 \xi_1^2}{2\lambda^2} - \frac{\tan(\varphi)}{6\lambda^2}\xi_1^3 \tag{22a}$$

$$y_2(\xi_2) = A_2 + B_2\xi_2 + C_2 \cos(\lambda\xi_2) + D_2 \sin(\lambda\xi_2) + \frac{\bar{q}_1 - \alpha \tan(\varphi)}{2\lambda^2}\bar{\xi}_2^2 - \frac{\tan(\varphi)}{6\lambda^2}\bar{\xi}_2^3 \tag{22b}$$

where the last two terms represent particular solutions. The eight constants in Eqs. (22a) and (22b) are determined by satisfying the boundary conditions in Eqs. (5a), (5b) and (14). Requiring that the transverse displacement in Eq. (22a) satisfies the boundary conditions at the leftmost end, Eq. (5a) we get the expression of the constants

$$A_1 = -\bar{q}_1/\lambda^2, \quad C_1 = \bar{q}_1/\lambda^2 \tag{23}$$

Requiring that the transverse displacement $w_2(\xi_2)$ in Eq. (22b) vanishes at the middle support, $\xi_2 = 0$, we obtain $C_2 = -A_2$. The equations in the unknowns B_1, D_1, A_2, B_2 and D_2 are

$$B_1\alpha + D_1 \sin(\lambda\alpha) = \frac{\tan(\varphi)}{6\lambda^2}\alpha^3 - \frac{\bar{q}_1\alpha^2}{2\lambda^2} + [1-\cos(\lambda\alpha)]\frac{\bar{q}_1}{\lambda^4}$$

$$B_2 + \lambda D_2 - B_1 - D_1\lambda\cos(\lambda\alpha) = \frac{\bar{q}_1 \sin(\lambda\alpha)}{\lambda^3} + \frac{\tan(\varphi)\alpha^2}{2\lambda^2} - \frac{\bar{q}_1\alpha}{\lambda^2}$$

$$\gamma B_1 + [\lambda^2\sin(\lambda\alpha) + \gamma\lambda\cos(\lambda\alpha)]D_1 + A_2\lambda^2 = \gamma\left[\frac{\bar{q}_1}{\lambda^3}\sin(\lambda\alpha) + \frac{\tan(\varphi)}{2\lambda^2}\alpha^2 + \frac{\bar{q}_1\alpha}{\lambda^2}\right] - \frac{\bar{q}_1}{\lambda^2}\cos(\lambda\alpha)$$

$$[1-\cos(\lambda\beta)]A_2 + B_2\beta + D_2\sin(\lambda\beta) = \frac{\tan(\varphi)}{6\lambda^2}\beta^3 + \frac{\alpha\tan(\varphi) - \bar{q}_1}{2\lambda^2}\beta^2$$

$$A_2\lambda^2\cos(\lambda\beta) - D_2\lambda^2\sin(\lambda\beta) = \frac{\tan(\varphi)}{6\lambda^2}\beta + \frac{\alpha\tan(\varphi) - \bar{q}_1}{2\lambda^2} \tag{24}$$

The solution for Eq. (24) is not reproduced here for the sake of brevity. If the determinant

$$\Delta = \begin{vmatrix} \alpha & \sin(\lambda\alpha) & 0 & 0 & 0 \\ -1 & -\lambda\cos(\lambda\alpha) & 0 & 1 & \lambda \\ \gamma & \gamma\lambda\cos(\lambda\alpha) + \gamma^2\sin(\lambda\alpha) & \lambda^2 & 0 & 0 \\ 0 & 0 & [1-\cos(\lambda\beta)] & \beta & \sin(\lambda\beta) \\ 0 & 0 & \lambda^2\cos(\lambda\beta) & 0 & -\lambda^2\sin(\lambda\beta) \end{vmatrix} \tag{25}$$

differs from zero, the response is unique. Condition $\Delta = 0$ indicates buckling, with minimum buckling load denoted by λ_1. We set $\lambda < \lambda_1$ and investigate the response.

The displacement functions are represented in Fig. 4a–d for different values of the misplacement δ and several values of the applied load λ. In Fig. 4a, λ is fixed at $0.25\lambda_1$; the effect of the misplacement is observed to be insignificant. In Fig. 4b, the applied axial load λ is set to the 43.5% of the classical buckling load λ_1. We observe that the misplacements cause a significant alteration in the response pattern. In the ideal structure, without a misplacement, the displacement function, shown by the solid line, has two *zeroes* (denoted by ξ_{10} and ξ_{20}) in addition to the end points and intermediate support location. Yet, with misplacements constituting 1% or more of the length L, these additional zeroes disappear. The maximum non-dimensional displacement in the ideal system, in the first span, equals 0.0008; whereas for 1% misplacement, the maximum displacement equals 0.0016, constituting a two-fold increase; for 2% misplacement this enhancement ratio constitutes 2.65. Finally, for 3% misplacement the increase is 5.5-fold. For the value of the axial load $\lambda = 0.47\lambda_1$, the presence of an additional *zero* is introduced in the displacement function of the imperfect column in one of the spans. On the other hand, the perfect structure does not have any zeroes other than in the location of the supports. An imperfect column subjected to axial load equal to 75% of the classical buckling load shows a behavior that is only slightly different from the ideal system (Fig. 4d). Analogous behavior is also recorded for $\lambda = 0.25\lambda_1$ (Fig. 4a). An important difference between Fig. 4a and d lies in the following: In Fig. 4a, the response of the imperfect structure exceeds that of the ideal one; whereas in Fig. 4d, the behavior is opposite.

The maximum values of the peak response ratio η defined as

$$\eta_j = \frac{\max_{\xi_i} |y_j^{\text{imperfect}}(\xi_j)|}{\max_{\xi_i} |y_j^{\text{ideal}}(\xi_j)|}, \quad j = 1, 2 \tag{26}$$

with $|\circ|$ representing the absolute value operator, are also reported in Fig. 4a–d, j is the span number. The maximum peak ratio occurs in the first span for $\lambda = 0.43\lambda_1$ and in the second span for $\lambda = 0.435\lambda_1$.

Whereas for the anti-symmetric loading, the peak response ratio may reach 5.5, in the case of symmetric loaded, or non-symmetric, partially loaded column, the misplacement of 3% provides only about 30% increment of the response compared with the ideal system. Still, such an effect cannot be overlooked, for percentagewise increase of the response is about 10 times greater than the percentagewise imperfection.

The peak response ratios η_j defined in Eq. (26) are depicted for the case of anti-symmetric transverse load $\bar{q}(\xi)$ in Fig. 5, as functions of the non-dimensional load ratio λ/λ_1. The global abscissa ξ is identified by means of the relation

$$\xi = \xi_1 + \langle \xi_1 - \alpha \rangle^0 \xi_2$$

$$\langle \xi_1 - \alpha \rangle^0 = 1 \quad \text{if } \xi_1 \geq \alpha$$

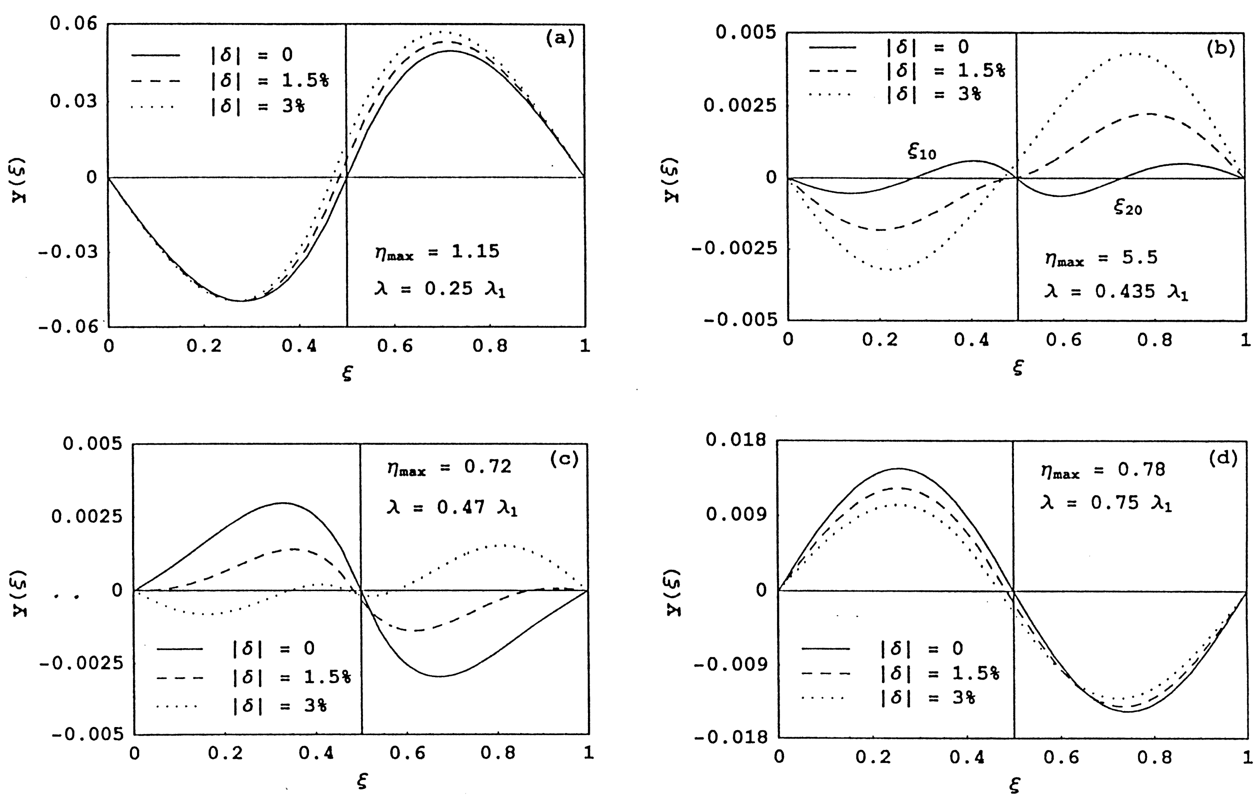

Fig. 4. Displacement functions $y(\xi)$ for different values of misplacements δ and axial load λ subjected to anti-symmetric transverse load with $\bar{q}_1 = -\bar{q}_2 = 8$.

$$\langle \xi_1 - \alpha \rangle^0 = 0 \quad \text{elsewhere} \tag{27}$$

where $\langle \cdot \rangle^0$ in Eq. (27) is denoted as singularity function. It is seen that the peak response ratio reaches values close to $\eta_2 = 15$ in the neighborhood of the non-dimensional ratio $\lambda/\lambda_1 = 0.43$. The effect of the misplaced middle support may lead to peak response ratios smaller than unity for values of the ratio $\lambda/\lambda_1 \geq 0.455$. Thus, the misplacement may have a beneficial effect on the response of the system.

4. Effect of misplacement on response in presence of axial, eccentric and transverse loads

Consider now a two-span column that is subjected to an axial load P applied with an eccentricity e from the column's axis as well as to a transverse load that is anti-symmetric with respect to the mid-span as in Eqs. (19a) and (19b). The effect of the eccentric load is equivalent to an external bending moment $M_e = Pe$ applied at the outer supports A and C. The differential equation governing the transverse displacement of the column are given by Eqs. (7) and (8). The boundary conditions at the outermost supports A and C, respectively, read

$$y_1(0) = 0 \quad \text{and} \quad \left. \frac{d^2 y_1(\xi_1)}{d\xi_1^2} \right|_0 = \lambda^2 \varepsilon \tag{28a}$$

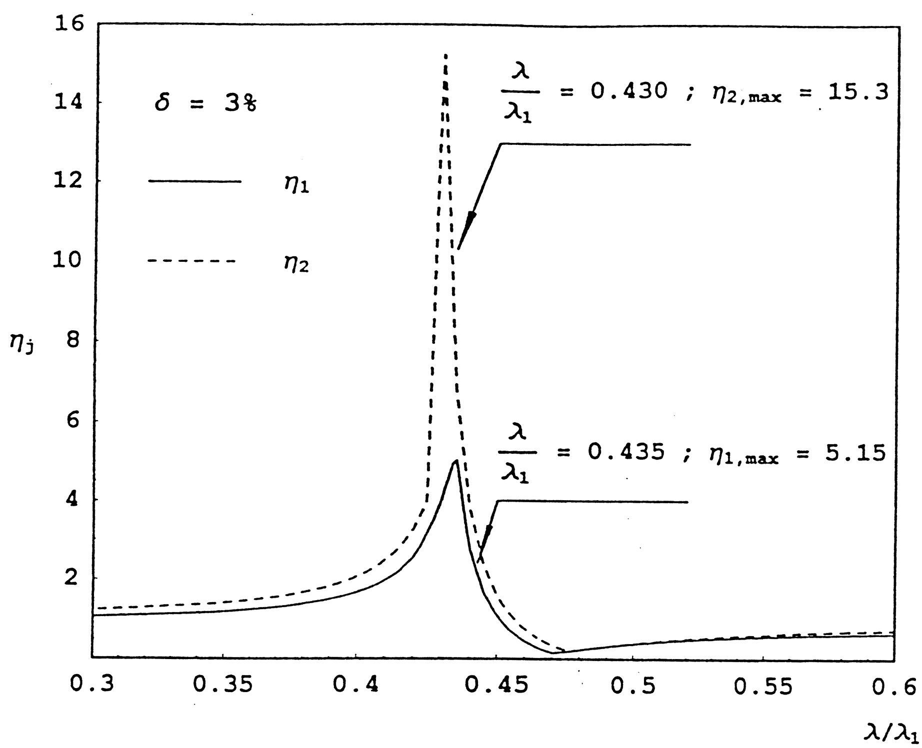

Fig. 5. Peak response ratios $\eta_1(\lambda/\lambda_1)$ and $\eta_2(\lambda/\lambda_1)$ for anti-symmetrically loaded column with $\bar{q}_1 = -\bar{q}_2 = 8$.

$$y_2(\beta) = 0 \quad \text{and} \quad \left.\frac{d^2 y_2(\xi_2)}{d\xi_2^2}\right|_\beta = \lambda^2 \varepsilon,\ \varepsilon = e/L \tag{28b}$$

The continuity conditions at support B are represented by the non-dimensional counterpart of Eqs. (5a), (5b) and (14). Imposing the boundary conditions at the leftmost support A on the general solution of Eq. (5a), we get the values of the constants A_1 and C_1

$$A_1 = C_1 = \varepsilon \tag{29}$$

leading to the solution of Eq. (5a) in terms of the remaining two constants B_1 and D_1

$$y_1(\xi) = \varepsilon\left[1 - \cos(\lambda\xi_1)\right] + B_1\xi_1 + D_1 \sin(\lambda\xi_1) + \frac{\bar{q}_1}{2\lambda^2}\xi_1^2 - \frac{\tan(\varphi)}{6\lambda^2}\xi_1^3 \tag{30}$$

The general expression of the non-dimensional transverse displacement $y_2(\xi)$ is

$$y_2(\xi) = A_2 + B_2\xi_2 + C_2 \cos(\lambda\xi) + D_2 \sin(\bar{\lambda}\xi) + \frac{\bar{q}_1 - \alpha \tan(\varphi)}{2\lambda^2}\xi_2^2 - \frac{\tan(\varphi)}{6\lambda^2}\xi_2^3 \tag{31}$$

Satisfying the continuity condition in Eq. (5b), we get $A_2 = -C_2$. Requiring that the functions in Eqs. (30) and (31) satisfy the remaining boundary and continuity conditions, we get five equations in the five unknowns B_1, D_1, A_2, B_2, C_2 and D_2

$$B_1\alpha + D_1 \sin(\lambda\alpha) = \varepsilon\left[\cos(\lambda\alpha) - 1\right] + \frac{\tan(\varphi)}{6\lambda^2}\alpha^3 - \frac{\bar{q}_1\alpha^2}{2\lambda^2} + \frac{\bar{q}_1}{\lambda^4}\left[1 - \cos(\lambda\alpha)\right]$$

$$B_2 + D_2\lambda - B_1 - \lambda D_1 \cos(\lambda\alpha) = -\lambda\varepsilon \sin(\lambda\alpha) + \frac{\bar{q}_1 \sin(\lambda\alpha)}{\lambda^3} + \frac{\tan(\varphi)\alpha^2}{2\lambda^2} - \frac{\bar{q}_1\alpha}{\lambda^2}$$

$$\begin{aligned}&\lambda^2 A_2 + \gamma B_1 + D_1\left[\lambda^2 \sin(\lambda\alpha) + \lambda\gamma \cos(\lambda\alpha)\right]\\ &\quad = \lambda^2\varepsilon \cos(\lambda\alpha) - \lambda\gamma\varepsilon \sin(\lambda\alpha) + \gamma\left[\frac{\bar{q}_1}{\lambda^3}\sin(\lambda\alpha) + \frac{\tan(\varphi)}{2\lambda^2} + \frac{\bar{q}_1\alpha}{\lambda^2}\right] - \frac{\bar{q}_1}{\lambda^2}\cos(\lambda\alpha)\end{aligned}$$

$$\left[1 - \cos(\lambda\beta)\right]A_2 + B_2\beta + D_2 \sin(\lambda\beta) = \frac{\tan(\varphi)}{6\lambda^2}\beta^3 + \frac{\alpha \tan(\varphi) - \bar{q}_1}{2\lambda^2}\beta^2$$

$$\lambda^2 A_2 \cos(\lambda\beta) - D_2\lambda^2 \sin(\lambda\beta) = \varepsilon\lambda^2 + \frac{\tan(\varphi)}{6\lambda^2}\beta + \frac{\alpha \tan(\varphi) - \bar{q}_1}{2\lambda^2} \tag{32}$$

The solution of the system of equations in Eq. (32) is straightforward and it will not be reproduced here for the sake of brevity. The effect of the misplaced support on the displacement response is deduced from Fig. 6a–d. In Fig. 6a, the axial load has been set at $\lambda = 0.43\lambda_1$. It is seen that the effect of the misplacement is insignificant. In Fig. 6b, we observe some qualitative magnification of the response. Namely, the response function of the perfect structure has an additional zero in the second span. Its counterpart, for $\delta = 0.03$, does not have an additional zero. We conclude that the effect of the misplacement is qualitative by change the character of the response. Observing the patterns in Fig. 6c and d, we note that increments in the applied axial load cause the additional node in the transverse response to move towards the external support A.

The effect of the misplacement on the maximum response of the column is deduced from Fig. 7a and b where the peak response ratios η_j, introduced in Eq. (26), are plotted against λ/λ_1 for various values of the misplacement δ. Fig. 7a is associated with the maximum peak ratio in the first span, whereas Fig. 7b is portraying the peak response ratio in the second span. The response of the imperfect structure is about 2.35 times greater than its counterpart in the ideal system, in the first span. On the other hand, the examination of Fig. 6b yields that in the second span the peak response ratio η_2 is over 2.9 for $\lambda = 0.64\lambda_1$.

In order to study further the effect of the misplacement δ over the response of the column, we introduce the following non-dimensional ratios (Fig. 8)

$$\gamma_p = \left(\frac{\max\limits_{\xi_2} |y_2(\xi_2)|}{\max\limits_{\xi_1} |y_1(\xi_1)|} \right)_{\text{perfect}} \quad ; \quad \gamma_i = \left(\frac{\max\limits_{\xi_2} |y_2(\xi_2)|}{\max\limits_{\xi_1} |y_1(\xi_1)|} \right)_{\text{imperfect}} \tag{33}$$

Note that the ratios defined in Eq. (33) differ from Eq. (26); γ_p and γ_i represent the degree of asymmetry in the response. The value λ was fixed at $0.9\lambda_1$. We observe that the response of the column is highly

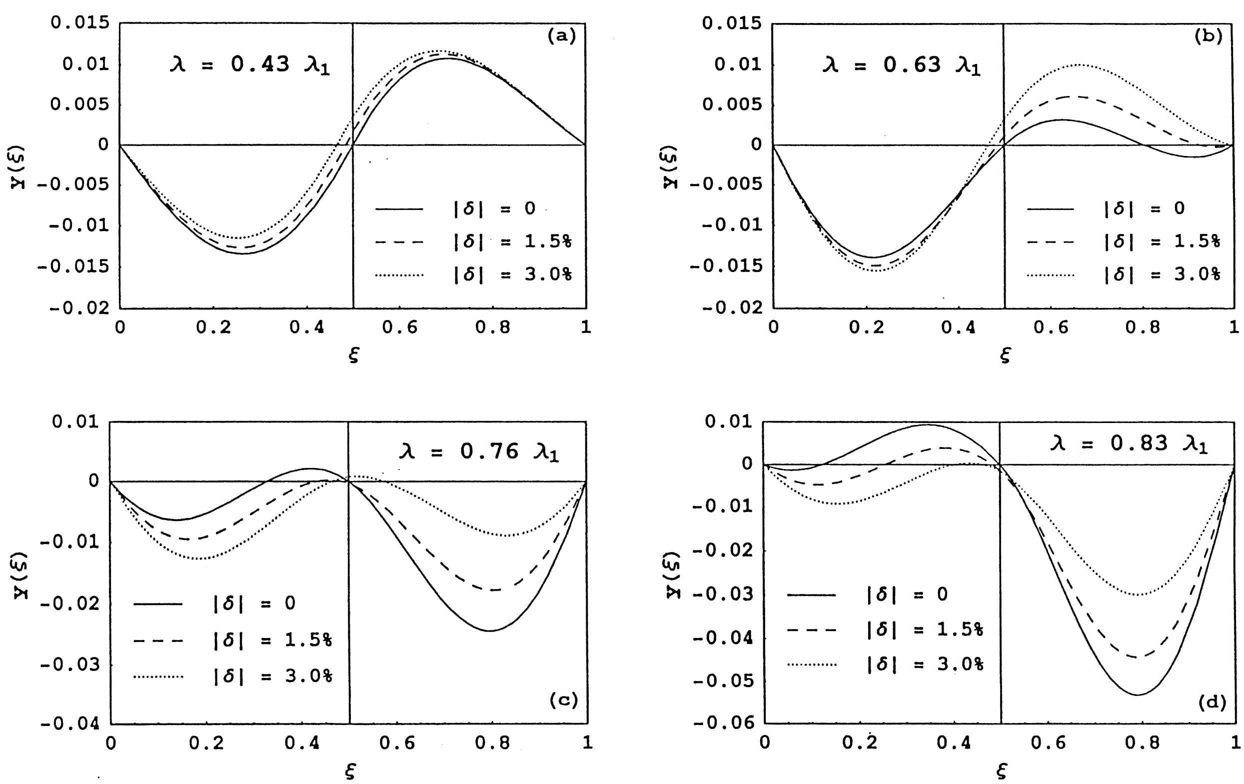

Fig. 6. Displacement function $y(\xi)$ for different misplacements δ and applied axial load for the column subjected to axially eccentric and transverse load with $\bar{q}_1 = -\bar{q}_2 = 4$ and $\varepsilon = 0.015$.

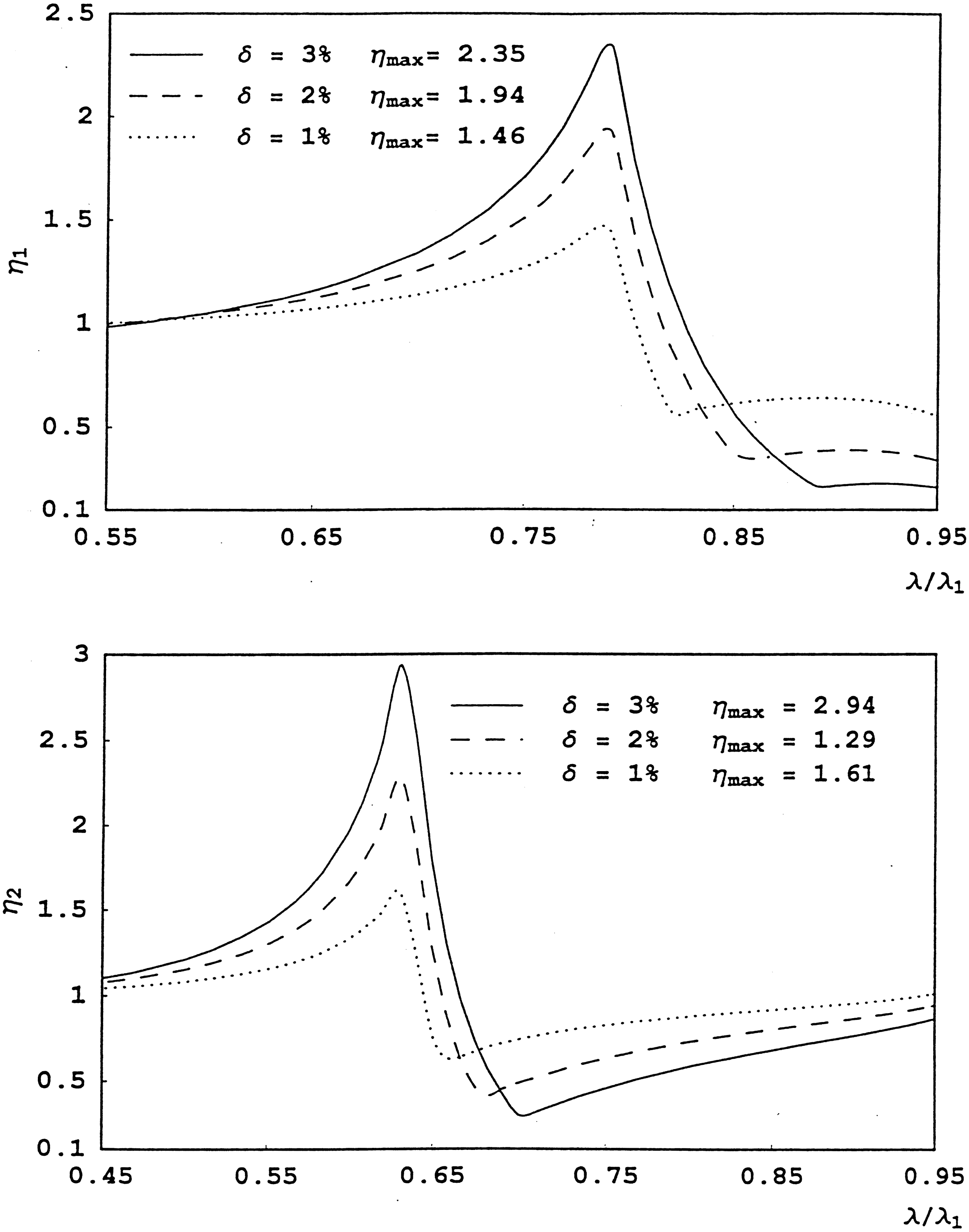

Fig. 7. Peak response ratios $\eta_1(\lambda/\lambda_1)$ and $\eta_2(\lambda/\lambda_1)$ for eccentrically and anti-symmetrically loaded column with $\bar{q}_1 = -\bar{q}_2 = 4$ and $\varepsilon = 0.015$.

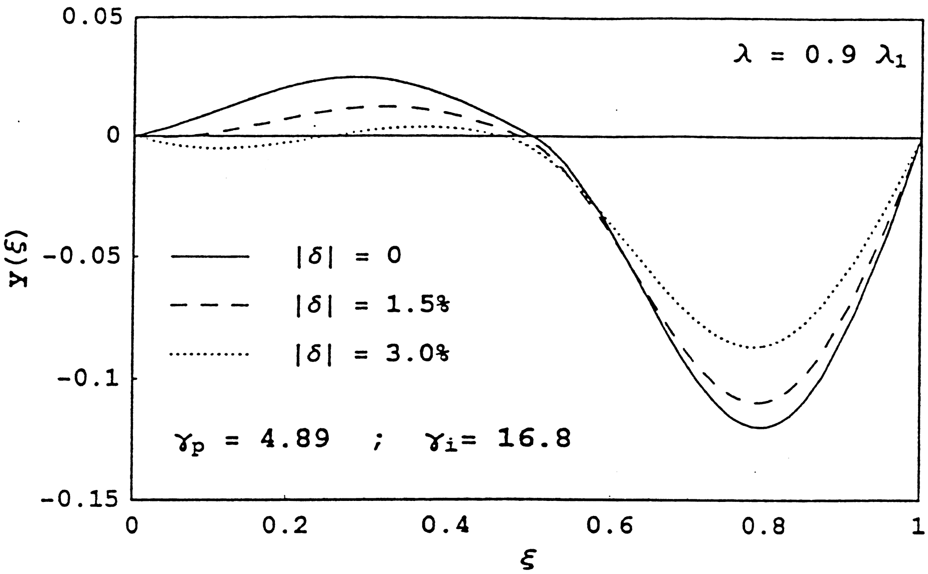

Fig. 8. Displacement function $y(\xi)$ with $\lambda = 0.9\lambda_1$ for the column subjected to axially eccentric and transverse load $\bar{q}_1 = -\bar{q}_2 = 4$ and $\varepsilon = 0.015$.

concentrated in the second span, whereas the first span experiences a nearly vanishing transverse displacement. The maximum response in the second span is 4.89 times than that in the first span, for the perfect system. Yet, the second span's maximum response turns out to be 16.8 times greater than the displacement in the first one when the misplacement $\delta = 0.03$; the asymmetry in the responses was increased over three times. Thus, we establish a localization pattern in the transverse response of the column.

5. Conclusions

In this study, we investigate the effect of misplacement in the location of the intermediate support on the response of a two-span column that is both axially and transversally loaded. The aim is to determine whether a slight disorder may lead to a large change in the response in contrast to the perfect structure. The investigation has shown significant influence of the misplacements in the transverse displacement of the column. Some configurations of the external transverse load, combined with the misplacements, cause significant change in both the qualitative and the quantitative dependence including a localization of the response. These conclusions were derived for a model structure, with attendant analytical simplicity. Yet rich variety of behaviors was uncovered. This work and its companion (Elishakoff and Zingales, 2000) were inspired when the second author was reading the Rabbi Arieh Kaplan's translation (1981) of the Torah ("the Bible"): "[All the beams] must be exactly next to each other on the bottom. [Every pair] shall also be [joined] together evenly on top with a [square] ring. This shall also be done with the two [beams] on the top corners," (Exodus, 26:24). The considerations pertinent to this study followed from the idea, that the Tabernacle would have been subjected to different loading conditions, during the periods of its enactment, daily services and transportation periods. Hence the spacing "evenly' (i.e. lack of misplacements) would lead to the lack of the localized response. Hence the conclusion was made by the second author that the response, rather than the mode shapes, was important quantity to investigate.

The companion study (Elishakoff and Zingales, 2000) in the vibration context, is underway and will be reported elsewhere.

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